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A Clifford Theory for Graded Coalgebras: Applications

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1. INTRODUCTION

The notion of graded coalgebras rarely appears in the literature on coalgebras (e.g., [1, 13]). The only case considered in these basic references is the case of a positively graded coalgebra. We introduced the general notion of G-graded coalgebras in [11]. For such a G-graded coalgebra, $C = \oplus_{\sigma \in G} C_\sigma$, it has been proved in loc. cit. that C_1 is a coalgebra and the natural projection π_1: $C \to C_1$ is a coalgebra morphism. We consider three related categories, the category gr^C of graded right C-comodules, the category $\mathbf{M}^C$ of right C-comodules, and the category $\mathbf{M}^{C_1}$ of right C_1-comodules. In [11], we have studied in some depth the connection between the three categories mentioned. It is a natural question to ask about the properties of $M \in gr^C$ that are preserved under the application of the forgetful functor U: $gr^C \to \mathbf{M}^C$, or more generally how these properties transform. For example, it is proved that if Q is injective in gr^C then $U(Q)$ is injective in $\mathbf{M}^C$, cf. Corollary 3.4. This philosophy also prompted the study of the structure of $U(\Sigma)$ for the simple object Σ of gr^C. This study is in fact the coalgebra equivalent of the Clifford Theory for graded rings as it has been established in [2, 3, 7] and several references in the latter. The graded simple comodules have a particular structure as C_1-comodules. As the main result of this paper we list Theorem 3.6, Theorem 4.2, and its corollaries. In order to arrive at this structure theory we have to develop in the first section (Sect. 3) some technical prerequisites that will be used in following sections.

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2. Notation and Preliminaries

Throughout this paper we work over a fixed ground field k. We shall freely use a version of "sigma notation" (see Sweedler's book [13] or Abe's book [1]). For other unexplained notation see [5, 12]. $\mathbf{M}^C$ (resp. ${}^C\mathbf{M}$) denote the category of right (resp. left) C-comodules. It is well known that $\mathbf{M}^C$ is an abelian category (see [13, 1], etc.). In fact $\mathbf{M}^C$ is a Grothendieck category.

Let G be a group with $1 \in G$ the identity element of G. Recall that a coalgebra C is called a G-graded coalgebra [11] if C is a direct sum of k-spaces $C = \oplus_{\sigma \in G} C_\sigma$, such that $\Delta(C_\sigma) \subseteq \Sigma_{\lambda\mu=\sigma} C_\lambda \otimes C_\mu$ for any $\sigma \in G$ and $\varepsilon(C_\sigma) = 0$ for any $\sigma \neq 1$. If M is a right C-comodule, then M is called a G-graded comodule over C if M admits a decomposition as a direct sum of k-spaces $M = \oplus_{\sigma \in G} M_\sigma$ such that $\rho_M(M_\sigma) \subseteq \Sigma_{\lambda\mu=\sigma} M_\lambda \otimes C_\mu$ for any $\sigma \in G$. Associated to any G-graded coalgebra $C = \oplus_{\sigma \in G} C_\sigma$ we have the category gr^C of all right graded C-comodules. In this category if M and N are two objects, then a morphism from M to N is a morphism $f \in \mathrm{Com}_C(M, N)$ satisfying that $f(M_\sigma) \subseteq N_\sigma$ for all $\sigma \in G$. We denote $\mathrm{Coend}_{gr^C}(M) = \mathrm{Com}_{gr^C}(M, M)$.

Let $\pi\colon C \to C_1$ be the canonical projection. By [11] we may define a structure of coalgebra on C_1, $(C_1, \Delta_1, \varepsilon)$, and $\pi\colon C \to C_1$ is a morphism of coalgebras. Moreover, if $M = \oplus_{\sigma \in G} M_\sigma$ is a right graded C-comodule, then for any $\sigma \in G$, M_σ is a right C_1-comodule via the canonical map $u^M_{\sigma,1}\colon M_\sigma \to M_\sigma \otimes C_1$ (i.e., $u^M_{\sigma,1}(m) = \Sigma_{(m)} m_0 \otimes \pi(m_1)$, for any $m \in M_\sigma$).

Let C be an arbitrary coalgebra, M a right C-comodule, and N a left C-comodule. The *cotensor product* $M \square_C N$ is the kernel of the k-map $\rho_M \otimes 1 - 1 \otimes \rho_N\colon M \otimes N \to M \otimes C \otimes N$.

Let $C = \oplus_{\sigma \in G} C_\sigma$ be a G-graded coalgebra and let $M \in \mathbf{M}^{C_1}$ be a right C_1-comodule. Since C has in the natural way (via the morphism π) a structure of (C_1, C)-bicomodule, we can consider $M \square_{C_1} C \in \mathbf{M}^C$. Since $C = \oplus_{\sigma \in G} C_\sigma$ and C_σ is a left C_1-comodule (see [11, Corollary 3.1]) (in fact C_σ is a (C_1, C_1)-bicomodule), for any $\sigma \in G$, then $M \square_{C_1} C = \oplus_{\sigma \in G}(M \square_{C_1} C_\sigma)$. Thus $M \square_{C_1} C$ is a right G-graded C-comodule. Clearly, in this way, we obtain a left exact functor $- \square_{C_1} C\colon \mathbf{M}^{C_1} \to gr^C$, $M \to M \square_{C_1} C$, $M \in M^{C_1}$. This functor is called the *coinduced functor*.

For any $\sigma \in G$, the functor $(-)_\sigma\colon gr^C \to \mathbf{M}^{C_\sigma}$, defined by $M \to M_\sigma$ for any $M \in gr^C$, is exact. By [11, Proposition 4.1], the functor $- \square_{C_1} C\colon \mathbf{M}^{C_1} \to gr^C$ is a right adjoint of the functor $(-)_1\colon gr^C \to \mathbf{M}^{C_1}$ and $(-)_1(- \square_{C_1} C) \simeq 1_{\mathbf{M}^{C_1}}$. The functors $- \square_{C_1} C$ and $(-)_1$ play a very important role in the study of connections between the categories gr^C and $\mathbf{M}^{C_1}$.

Let $M = \oplus_{\sigma \in G} M_\sigma$ be an object in gr^C and $\sigma \in G$. We can define another G-graded comodule denoted by $M(\sigma)$: as C-comodule $M(\sigma)$ coincides with M, i.e., $M(\sigma) = M$ and $\rho_{M(\sigma)} = \rho_M$, but the grading of

$M(\sigma)$ is given by the equality $M(\sigma)_\lambda = M_{\sigma\lambda}$ for any $\lambda \in G$. $M(\sigma)$ is called the *σ-suspension* of M.

If $C = \oplus_{\sigma \in G} C_\sigma$ is an arbitrary coalgebra, then we can define the dual algebra, $C^* = \mathrm{Hom}_k(C, k)$ with convolution as multiplication (cf. [13]). Let $\rho_M: M \to M \otimes C$ be a right C-comodule. Then M has a natural structure of left C^*-module. Let $C = \oplus_{\sigma \in G} C_\sigma$ be a G-graded coalgebra. For any $\sigma \in G$ we put $R_\sigma = \{f \in C^* | f(C_\tau) = 0$ for all $\tau \neq \sigma\}$. We define $R = \Sigma_{\sigma \in G} R_\sigma = \oplus_{\sigma \in G} R_\sigma$; by Proposition 6.1 of [11] R is a G-graded ring with $\varepsilon: C \to k$ the identity map.

If $C = \oplus_{\sigma \in G} C_\sigma$ is a G-graded coalgebra and $M = \oplus_{\sigma \in G} M_\sigma$ together with $\rho_M: M \to M \otimes C$ is a right graded C-comodule. We define a left graded R-module, $\overline{M}$, in the following way: $\overline{M} = M$ as k-space but for any $\sigma \in G$ we put $\overline{M}_\sigma = M_{\sigma^{-1}}$. Now, if $f \in R_\sigma$ and $m \in \overline{M}_\tau = M_{\tau^{-1}}$, then we have $fm = \Sigma_{(m)} m_0 f(m_1)$, where $\rho_M(m) = \Sigma_{(m)} m_0 \otimes m_1$. Denoting the category of left graded R-modules by $R - gr$, we have obtained a functor $H: gr^C \to R - gr$, $H(M) = \overline{M}$.

Let us denote the full subcategory of $R - gr$ of all gr-rational modules by $\mathrm{Rat}(R - gr)$. It is easy to see that $\mathrm{Rat}(R - gr)$ is a closed subcategory of $R - gr$. Moreover $\mathrm{Rat}(R - gr)$ is rigid (or a G-invariant closed subcategory of $R - gr$; i.e., for any $M \in \mathrm{Rat}(R - gr)$ the σ-suspension $M(\sigma) \in \mathrm{Rat}(R - gr)$ for every $\sigma \in G$ (see [10]). Since $\mathrm{Rat}(R - gr)$ is closed under direct sums and quotients, then for any $M \in R - gr$ there exists $\mathrm{Rat}(M)$, the largest graded submodule belonging to $\mathrm{Rat}(R - gr)$. By [11, theorem 6.3], H gives an isomorphism between the categories gr^C and $\mathrm{Rat}(R - gr)$.

3. The Subcategories $(gr^C)_\sigma$, $\sigma \in G$: Applications to the Structure of Graded Simple Comodules

If $C = \oplus_{\sigma \in G} C_\sigma$ is a G-graded coalgebra, in the study of simple objects, injective objects, etc. from the category gr^C, the subcategories $(gr^C)_\sigma$ are very important, where $(gr^C)_\sigma = \{M \in gr^C | M = \oplus_{\lambda \in G} M_\lambda$ such that $M_\sigma = 0\}$.

First some easy properties of these subcategories are noted.

Proposition 3.1. *For any $\sigma \in G$, $(gr^C)_\sigma$ is a localizing subcategory of gr^C (cf. [6]). Moreover $(gr^C)_\sigma$ is closed under direct product.*

Since $(gr^C)_\sigma$ is a localizing subcategory of gr^C, then for $M \in gr^C$ we can consider the greatest subobject $t_\sigma(M)$ of M belonging to $(gr^C)_\sigma$. If $t_\sigma(M) = 0$ then M is said to be σ-faithful. The mapping $M \to t_\sigma(M)$ defines a left exact functor $t_\sigma: gr^C \to gr^C$. We note that M is σ-faithful in the category gr^C if and only if $\overline{M} = H(M)$ is σ^{-1}-faithful in the category $R - gr$ (see [9]).

PROPOSITION 3.2. *Let $M = \oplus_{\lambda \in G} M_\lambda$ be an object in gr^C. For any $\sigma \in G$ there exists a morphism $\mu(M)$: $M \to (M_\sigma \square_{C_1} C)(\sigma^{-1})$ with the following properties*:

(i) Ker $\mu(M)$ *and* Coker $\mu(M)$ *belong to* $(gr^C)_\sigma$.

(ii) $M_\sigma \square_{C_1} C$ *is 1-faithful.*

(iii) Im $\mu(M)$ *is an essential subobject of* $M_\sigma \square_{C_1} C$.

Proof. Since $M(\sigma)_1 = M_\sigma$, we can assume $\sigma = 1$, i.e., $M(1) = M$. Since the functor $- \square_{C_1} C$: $\mathbf{M}^{C_1} \to gr^C$ is the right adjoint of the functor $(-)_1$ [11, Proposition 4.1], then we have the canonical functorial isomorphism

$$\mathrm{Com}_{C_1}(M_1, N) \underset{\psi(M,N)}{\overset{\varphi(M,N)}{\rightleftarrows}} \mathrm{Com}_{gr^C}(M, N \square_{C_1} C) \tag{1}$$

(ψ the inverse of φ). We put $N = M_1$ and $\mu(M) = \varphi(M, M_1)(1_{M_1})$; $\mu(M)$: $M \to M_1 \square_{C_1} C$. In fact, following [11], if $m \in M$ is an homogenous element and $\rho_M(m) = \Sigma_{(m)} m_0 \otimes m_1$, where $m_0 \in M$, $m_1 \in C$ are homogeneous elements then $\mu(M)(m) = \Sigma_{\deg m_0 = 1} m_0 \otimes m_1$. Since $\varepsilon(C_\sigma) = 0$ for $\sigma \neq 1$, we observe that $\mu(M)_1$: $M_1 \to (M_1 \square_{C_1} C)_1$ is an isomorphism. The exact sequence

$$0 \to \mathrm{Ker}\, \mu(M) \to M \to M_1 \square_{C_1} C \to \mathrm{Coker}\, \mu(M) \to 0$$

yields that $(\mathrm{Ker}\, \mu(M))_1 = (\mathrm{Coker}\, \mu(M))_1 = 0$. Thus, Ker $\mu(M)$ and Coker $\mu(M)$ belong to $(gr^C)_\sigma$.

(ii) We denote by $X = t_1(M_1 \square_{C_1} C)$. Hence $X_1 = 0$. If in (1) we put $M = X$ and $N = M_1$ it follows that the inclusion morphism i: $X \to M_1 \square_{C_1} C$ is zero. Hence $X = 0$.

(iii) This follows from (i) and (ii).

Using the previous proposition we may determine the complete structure of the injective objects in gr^C. Since the category gr^C is a locally finite category, gr^C is a locally noetherian category, i.e., gr^C has a family of noetherian generators (see [12]). By [12, Proposition V.4.5], if $Q \in gr^C$ is an injective object, then $Q = \oplus_{i \in I} Q_i$, where the Q_i are injective indecomposable objects; moreover this decomposition is unique. Thus, we only have to consider the indecomposable injective objects of gr^C.

The main result about an indecomposable injective object of the category gr^C can now be obtained.

THEOREM 3.3. *Let* $Q = \oplus_{\lambda \in G} Q_\lambda$ *be a nonzero indecomposable injective object in the category* gr^C. *The following assertions hold*:

(i) *There exists* $\sigma \in G$ *such that* Q *is* σ*-faithful.*

(ii) $Q \simeq (Q_\sigma \square_{C_1} C)(\sigma^{-1})$ *and* Q_σ *is an indecomposable injective object in the category* $\mathbf{M}^{C_1}$.

Proof. (i) Since gr^C is a locally finite category Q contains a simple object S. Since Q is indecomposable injective in gr^C, then $Q = E^g(S)$, where $E^g(S)$ denotes the injective envelope of S in the category gr^C. Assume that $S = \oplus_{\lambda \in G} S_\lambda$. Since $S \neq 0$, there exists $\sigma \in G$ such that $S_\sigma \neq 0$. If $t_\sigma(S) \neq 0$ and since S is simple it follows that $t_\sigma(S) = S$. Hence $S_\sigma = (t_\sigma(S))_\sigma = 0$, which is a contradiction. Therefore $t_\sigma(S) = 0$ and S is σ-faithful. Now, since Q is an essential extension of S, then $t_\sigma(Q) = 0$, i.e., Q is σ-faithful.

(ii) By Proposition 3.2 we have the morphism $\mu(Q)$: $Q \to (Q_\sigma \square_{C_1} C)(\sigma^{-1})$. Since Q is σ-faithful, $\mu(Q)$ is a monomorphism. Since $\operatorname{Im} \mu(Q)$ is essential in $(Q_\sigma \square_{C_1} C)(\sigma^{-1})$ and Q is injective, it follows that $\mu(Q)$ is an isomorphism.

We now prove that Q_σ is indecomposable injective in $\mathbf{M}^{C_1}$. From the exact sequence $0 \to Q_\sigma \to E(Q_\sigma)$ and using the fact that the cotensor product is left exact, we obtain the exact sequence $0 \to Q_\sigma \square_{C_1} C \to E(Q_\sigma) \square_{C_1} C$. Since $Q(\sigma) \cong Q_\sigma \square_{C_1} C$ is injective, then there exists an object $X \in gr^C$ such that $E(Q_\sigma) \square_{C_1} C \cong X \oplus (Q_\sigma \square_{C_1} C)$. Applying the functor $(-)_1$, we get $E(Q_\sigma) \cong X_1 \oplus Q_\sigma$. Hence $X_1 = 0$ and $X \subseteq t_1(E(Q_\sigma) \square_{C_1} C)$. But $E(Q_\sigma) \square_{C_1} C$ is 1-faithful. Hence $X = 0$. Thus $Q_\sigma \square_{C_1} C \cong E(Q_\sigma) \square_{C_1} C$ and applying the same functor $(-)_1$, we obtain $Q_\sigma \cong E(Q_\sigma)$, i.e., Q_σ is injective in $\mathbf{M}^{C_1}$. On the other hand it is obvious that Q_σ is indecomposable.

COROLLARY 3.4. *Let* $Q \in gr^C$. Q *is injective in* gr^C *if and only if* Q *is injective in* $\mathbf{M}^C$.

Proof. Since Q is a direct sum of injective indecomposable objects in the category gr^C, we may assume that Q is indecomposable. In this case Theorem 3.3 yields $Q \cong (Q_\sigma \square_{C_1} C)(\sigma^{-1})$ for some $\sigma \in G$. Hence in the category $\mathbf{M}^C$ we have $Q \cong Q_\sigma \square_{C_1} C$. The functor $- \square_{C_1} C$ is the right adjoint of the functor $(-)_\pi$: $\mathbf{M}^C \to \mathbf{M}^{C_1}$, where π: $C \to C_1$ is the canonical projection (cf. [5]). Since the functor $(-)_\pi$ is exact and Q_σ is injective, $Q_\sigma \square_{C_1} C$ is injective in $\mathbf{M}^C$. Hence Q is injective in $\mathbf{M}^C$.

Conversely, assume that Q is injective in $\mathbf{M}^C$. We consider u: $M' \to M$ a subcomodule of M and f: $M' \to Q$ a morphism in the category gr^C. Since Q is injective in $\mathbf{M}^C$, then there exists a morphism g in the category $\mathbf{M}^C$ such that $g \circ u = f$.

We define the map $h: M \to Q$ in the following way. If $m \in M_\sigma$ is a homogeneous element, then we put $h(m) = g(m)_\sigma$, where $g(m)_\sigma$ is the homogeneous component of the element $g(m)$. Clearly we have $h \circ u = f$. It remains to show that h is also a morphism of C-comodules. Assume $m \in M_\sigma$. We have $\rho_M(m) \in \Sigma_{xy=\sigma} M_x \otimes C_y$. Thus $\rho_M(m) = \Sigma m_0 \otimes m_1$, where $m_0 \in M_x$ and $m_1 \in C_y$ with $xy = \sigma$. If we write $g(m) = g(m)_\sigma + \Sigma_{\lambda \neq \sigma} g(m)_\lambda = h(m) + \Sigma_{\lambda \neq \sigma} g(m)_\lambda$. From the equality $\rho_Q(g(m)) = (g \otimes 1)\rho_M(m)$, it follows that $\rho_Q(h(m)) + \Sigma_{\lambda \neq \sigma} \rho_Q(g(m)_\lambda) = \Sigma_{(m)} g(m_0) \otimes m_1 = \Sigma_{(m)} g(m_0)_x \otimes m_1 + \Sigma_{(m)} \Sigma_{x' \neq x} g(m_0)_{x'} \otimes m_1$. Hence $\rho_Q(h(m)) + \Sigma_{\lambda \neq \sigma} \rho_Q(g(m)_\lambda) = \Sigma_{(m)} h(m_0) \otimes m_1 + \Sigma_{(m)} \Sigma_{x' \neq x} g(m_0)_{x'} \otimes m_1$. This equality yields that $\rho_Q(h(m)) = \Sigma_{(m)} h(m_0) \otimes m_1$. Thus $\rho_Q \circ h = (h \otimes 1) \circ \rho_M$.

Since for any $\sigma \in G$, $(gr^C)_\sigma$ is a localizing subcategory of gr^C, then we can consider the quotient category $gr^C/(gr^C)_\sigma$ and we denote by

$$gr^C \underset{S_\sigma}{\overset{F_\sigma}{\rightleftarrows}} gr^C/(gr^C)_\sigma$$

the canonical functors (for detail concerning the quotient categories see [6]).

We consider the functors U: $\mathbf{M}^{C_1} \to gr^C/(gr^C)_\sigma$ given by $U = F_\sigma \circ T_{\sigma^{-1}} \circ (- \square_{C_1} C)$, where $T_{\sigma^{-1}}$ is the σ^{-1}-suspensional functor and V: $gr^C/(gr^C)_\sigma \to \mathbf{M}^{C_1}$ is given by $V = (-)_\sigma \circ S_\sigma$. Using Proposition 3.2 we obtain, by straightforward computation, that $V \circ U \simeq 1_{\mathbf{M}^{C_1}}$ and $U \circ V \simeq 1_{gr^C/(gr^C)_\sigma}$. In fact the proof is similar to the proof given in [9, Theorem 3.1]. Thus, we have obtained the following result

COROLLARY 3.5. *With the above notation, the functors U and V define an equivalence between the categories $\mathbf{M}^{C_1}$ and $gr^C/(gr^C)_\sigma$.*

Remark. The preceding result has been proved in [4, Corollary 2.7] using other methods. Let $C = \oplus_{\sigma \in G} C_\sigma$ be a G-graded coalgebra and $M \in gr^C$. Since the subcategory $(gr^C)_\sigma$ is closed under direct products, there exists a smallest subobject of M, say $s_\sigma(M)$, such that $M/s_\sigma(M) \in (gr^C)_\sigma$. In fact $s_\sigma(M) = \bigcap\{K | K \subseteq M$ a graded submodule such that $M/K \in (gr^C)_\sigma\}$. Now consider a morphism f: $M \to N$ in the category gr^C together with the sequence

$$s_\sigma(M) \xrightarrow{i} M \xrightarrow{f} N \xrightarrow{\pi} N/s_\sigma(N).$$

Since $M/\mathrm{Ker}(\pi \circ f) \subseteq N/s_\sigma(N)$ it follows that $M/\mathrm{Ker}(\pi \circ f) \in (gr^C)_\sigma$. But then $s_\sigma(M) \subseteq \mathrm{Ker}(\pi \circ f)$, which implies $\pi(f(s_\sigma(M))) = 0$ and $f(s_\sigma(M)) \subseteq s_\sigma(N)$. We have seen that the correspondence $M \to s_\sigma(M)$ defines a functor s_σ: $gr^C \to gr^C$.

Using this functor we may describe the structure of simple objects of gr^C.

THEOREM 3.6. *The following assertions hold*

(i) *If $M \in \mathbf{M}^{C_1}$ is a simple C_1-comodule, then $s_1(M \,\square_{C_1} C)$ is a graded simple C-comodule.*

(ii) *Let $\Sigma = \oplus_{\lambda \in G} \Sigma_\lambda$ be a simple object in gr^C. If $\Sigma_\sigma \neq 0$ ($\sigma \in G$), then Σ_σ is a simple C_1-comodule and $\Sigma \cong (s_1(\Sigma_\sigma \,\square_{C_1} C))(\sigma^{-1})$.*

Proof. (i) Since $M \,\square_{C_1} C$ is 1-faithful (Proposition 3.2), then $s_1(M \,\square_{C_1} C) \neq 0$. Let $Y \subseteq s_1(M \,\square_{C_1} C)$ be a nonzero subobject of $s_1(M \,\square_{C_1} C)$. From the isomorphism $\mathrm{Com}_{C_1}(Y_1, M) \cong \mathrm{Com}_{gr^C}(Y, M \,\square_{C_1} C)$ it follows that $Y_1 \neq 0$. Since $(M \,\square_{C_1} C)_1 = M \,\square_{C_1} C_1 \cong M$ and it is a simple C-comodule, then $Y_1 = (M \,\square_{C_1} C)_1$; hence $(M \,\square_{C_1} C/Y)_1 = 0$. Thus $M \,\square_{C_1} C/Y \in (gr^C)_1$ and therefore $s_1(M \,\square_{C_1} C) \subseteq Y$. Hence $Y = s_1(M \,\square_{C_1} C)$ and $s_1(M \,\square_{C_1} C)$ is a simple object in gr^C.

(ii) First note that $\Sigma(\sigma)$ is also a simple object in gr^C; thus we can assume $\sigma = 1$. Since $\Sigma_1 \neq 0$, then Σ is 1-faithful and by Proposition 3.2 it follows that the canonical morphism $\mu(\Sigma)$: $\Sigma \to \Sigma_1 \,\square_{C_1} C$ is injective. Since $\mathrm{Coker}\, \mu(\Sigma) \in (gr^C)_1$, then $s_1(\Sigma_1 \,\square_{C_1} C) \subseteq \mathrm{Im}\, \mu(\Sigma)$. But $\mathrm{Im}\, \mu(\Sigma) \cong \Sigma$ is simple which implies $s_1(\Sigma_1 \,\square_{C_1} C) = \mathrm{Im}\, \mu(\Sigma)$, i.e., $\Sigma \cong \Sigma_1 \,\square_{C_1} C$.

We prove now that Σ_1 is a simple C_1-comodule. If $0 \neq X \subseteq \Sigma_1$ is a nonzero C_1-subcomodule, then we have $s_1(X \,\square_{C_1} C) \subseteq s_1(\Sigma_1 \,\square_{C_1} C) \cong \Sigma$. Since $s_1(X \,\square_{C_1} C) \neq 0$, then $s_1(X \,\square_{C_1} C) = s_1(\Sigma_1 \,\square_{C_1} C)$ and hence $(s_1(X \,\square_{C_1} C))_1 = (s_1(\Sigma_1 \,\square_{C_1} C))_1$. But since $X \cong (s_1(X \,\square_{C_1} C))_1$ and $\Sigma_1 \cong (s_1(\Sigma_1 \,\square_{C_1} C))_1$, then $X = \Sigma_1$.

COROLLARY 3.7. *Let $\Sigma = \oplus_{\lambda \in G} \Sigma_\lambda$ be a simple object in gr^C. If $\Sigma_\sigma \neq 0$, then*

$$\mathrm{Coend}_{gr^C}(\Sigma) \cong \mathrm{Coend}_{C_1}(\Sigma_\sigma).$$

Proof. We define φ: $\mathrm{Coend}_{gr^C} \to \mathrm{Coend}_{C_1}(\Sigma_\sigma)$, $\varphi(f) = f|\Sigma_\sigma$ where $f \in \mathrm{Coend}_{gr^C}(\Sigma)$. Assume that $\varphi(f) = 0$. Since $\mu(\Sigma)$ is a functorial morphism, then we have the commutative diagram –

$$\begin{array}{ccccc} 0 & \longrightarrow & \Sigma & \xrightarrow{\mu(\Sigma)} & (\Sigma_\sigma \,\square_{C_1} C)(\sigma^{-1}) \\ & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f|\Sigma_\sigma \square_{C_1} C} \\ 0 & \longrightarrow & \Sigma & \xrightarrow{\mu(\Sigma)} & (\Sigma_\sigma \,\square_{C_1} C)(\sigma^{-1}) \end{array}$$

– Since $f|\Sigma_\sigma = 0$, then $\mu(\Sigma) \circ f = 0$ and hence $f = 0$. Thus φ is injective.

Let $h \in \mathrm{Coend}_{C_1}(\Sigma_\sigma)$. Then we have the morphism $h \square_{C_1} C$: $\Sigma_\sigma \square_{C_1} C \to \Sigma_\sigma \square_{C_1} C$. Since s_1 is a functor, by Theorem 3.1 we obtain a morphism f: $\Sigma(\sigma) \to \Sigma(\sigma)$ in gr^C such that the diagram

$$\begin{array}{ccccc} 0 & \longrightarrow & \Sigma(\sigma) & \longrightarrow & s_1(\Sigma_\sigma \square_{C_1} C) \\ & & \downarrow f & & \downarrow s_1(h \square_{C_1} C) \\ 0 & \longrightarrow & \Sigma(\sigma) & \longrightarrow & s_1(\Sigma_\sigma \square_{C_1} C) \end{array}$$

is commutative. Clearly this implies that $\varphi(f) = h$.

Recall that a G-graded coalgebra, $C = \oplus_{\sigma \in G} C_\sigma$, is said to be *strongly graded coalgebra* if for any $\sigma, \tau \in G$ the canonical morphisms $u^C_{\sigma,\tau}$: $C_{\sigma\tau} \to C_\sigma \otimes C_\tau$, $c \to \Sigma \pi_\sigma(c_1) \otimes \pi_\tau(c_2)$ are monomorphisms (see [11]). (Here π_σ: $C \to C_\sigma$ is the canonical projection for any $\sigma \in G$.) In [11, Theorem 5.4] it has been proved that C is strongly graded if and only if the induction functor $- \square_{C_1} C$: $\mathbf{M}^{C_1} \to gr^C$ is an equivalence of categories; equivalently the functor $(-)_1$: $gr^C \to \mathbf{M}^{C_1}$ is an equivalence of categories. Using the categories $(gr^C)_\sigma$, $\sigma \in G$, we can give a new characterization of strongly graded coalgebras.

PROPOSITION 3.8. *Let $C = \oplus_{\sigma \in G} C_\sigma$ be a G-graded coalgebra and $\sigma \in G$. C is strongly graded coalgebra if and only if $(gr^C)_\sigma = \{0\}$.*

Proof. ($\Rightarrow$) Let $M = \oplus_{\sigma \in G} M_\sigma$, $M \in (gr^C)_1$. Then $M_\sigma = 0$. By [11, Theorem 5.4], we have $M(\sigma) \cong M_\sigma \square_{C_1} C$. Hence $M(\sigma) = 0$, i.e., $M = 0$.

($\Leftarrow$) Clearly, $(gr^C)_\sigma = 0$ if and only if $(gr^C)_1 = 0$. Then we can assume that $\sigma = 1$. By Proposition 2.2, we have the canonical monomorphism $\mu(M)$: $M \to M_1 \square_{C_1} C$, $M \in gr^C$. Since $\mathrm{Coker}\, \mu(M) \in (gr^C)_1$, then $\mathrm{Coker}\, \mu(M) = 0$. Hence $\mu(M)$ is an isomorphism. Thus C is strongly graded, by [11, Theorem 5.4].

As mentioned before [11, Theorem 5.4] gives that a graded coalgebra C is strongly graded if and only if the canonical functor $(-)_1$: $gr^C \to \mathbf{M}^{C_1}$ defines an equivalence. The following question arises: if gr^C is equivalent (not necessarily canonically) to $\mathbf{M}^{C_1}$ does it follow that C is a strongly graded coalgebra?. In general the answer is no as the following example shows.

EXAMPLE. Let $(C, \Delta_C, \varepsilon_G)$ and $(D, \Delta_D, \varepsilon_D)$ be two coalgebras. We can define the direct sum of coalgebras $C \oplus D$ with comultiplication given by $\Delta_{C \oplus D}$: $C \oplus D \to (C \oplus D) \otimes (C \oplus D)$ where $\Delta_{C \oplus D}(c, d) = \Sigma_{(c)}(c_1, 0) \otimes (c_2, 0) + \Sigma_{(d)}(0, d_1) \otimes (0, d_2)$, $\Delta_C(c) = \Sigma_{(c)} c_1 \otimes c_2$, and $\Delta_D(d) = \Sigma_{(d)} d_1 \otimes d_2$. The counity $\varepsilon_{C \oplus D}$: $C \oplus D \to k$ is given by the equality $\varepsilon_{C \oplus D}(c, d) = \varepsilon_C(c) + \varepsilon_D(d)$. With these definitions the canonical injection maps $i_C: C \to C \oplus D$ (resp. $i_D: D \to C \oplus D$) with $i_C(c) = (c, 0)$ (resp. $i_D(d) = (0, d)$) are coalgebra morphisms.

The categories $\mathbf{M}^{C \oplus D}$ and $\mathbf{M}^C \times \mathbf{M}^D$ are equivalent. Indeed we can define the functors $F: \mathbf{M}^{C \oplus D} \to \mathbf{M}^C \times \mathbf{M}^D$; $F(M) = (M \,\square_{C \oplus D} C, M \,\square_{C \oplus D} D)$, where $M \in \mathbf{M}^{C \oplus D}$ and $G: \mathbf{M}^C \times \mathbf{M}^D \to \mathbf{M}^{C \oplus D}$, $G(X, Y) = X \oplus Y$, where $X \in \mathbf{M}^C$, $Y \in \mathbf{M}^D$ and we make the direct sum $X \oplus Y$ considering X (resp. Y) as a $C \oplus D$-comodule via the morphism i_C (resp. i_D). It is easy to see that $F \circ G \simeq 1_{\mathbf{M}^{C \oplus D}}$ and $G \circ F \simeq 1_{\mathbf{M}^C \times \mathbf{M}^D}$.

Now we take the group $\mathscr{G} = \{1, g\}$ with $g^2 = 1$ and let C be an arbitrary coalgebra. We define the $\mathscr{G}$-graded coalgebra D in the following way. Put $D_1 = C$ and $D_g = 0$. Clearly $gr^D \simeq \mathbf{M}^C \times \mathbf{M}^C \simeq \mathbf{M}^{C \oplus C}$. Assume that S is an infinite set and consider $C = k[S]$, the vector k-space with basis S, and comultiplication $\Delta: k[S] \to k[S] \otimes k[S]$, $\Delta(s) = s \otimes s$ for any $s \in S$, and $\varepsilon: k[S] \to k$, $\varepsilon(s) = 1$, $s \in S$. But $k[S] \oplus k[S] \cong k[S \cup S]$, where $S \cup S$ is the disjunt union. Since S is infinite, $S \cup S$ is cardinal equivalent to S and we have that $k[S] \oplus k[S] \cong k[S]$ as coalgebras. Therefore we have $gr^D \simeq \mathbf{M}^C$ in this particular case, where $D_1 = C = k[S]$. But it is obvious that D is not strongly graded.

For an abelian category $\mathscr{A}$, we denote by $\Omega_{\mathscr{A}}$ the set of all isomorphism classes of simple objects from $\mathscr{A}$, i.e., $\Omega_{\mathscr{A}} = \{[S] \mid S \text{ is a simple object in } \mathscr{A}\}$ and $[S] = \{S' \in \mathscr{A} \mid S' \cong S\}$. In the particular case $\mathscr{A} = \mathbf{M}^C$, where C is a coalgebra, we put $\Omega_C = \Omega_{\mathscr{A}}$.

COROLLARY 3.9. *Let $C = \oplus_{\sigma \in G} C_\sigma$ be a G-graded coalgebra. Assume that gr^C is equivalent to the category $\mathbf{M}^{C_1}$ (the equivalence of categories is not necessarily canonical). If $|\Omega_{C_1}| < \infty$, then C is a strongly graded coalgebra. ($|\Omega_{C_1}|$ denotes the cardinal of the set Ω_{C_1}.)*

Proof. Since gr^C is equivalent to $\mathbf{M}^{C_1}$, it follows that $|\Omega_{gr^C}| = |\Omega_{C_1}| < \infty$. We denote by $\Omega_{gr^C, 1}$ the set of all isomorphism classes of simples objects of gr^C which are 1-faithful. By Theorem 3.1 assertion (i) it follows that $|\Omega_{C_1}| \leq |\Omega_{gr^C, 1}|$. Since $|\Omega_{gr^C, 1}| \leq |\Omega_{gr^C}|$, then we have $|\Omega_{gr^C, 1}| = |\Omega_{gr^C}|$. If $(gr^C)_1 \neq \{0\}$, then $(gr^C)_1$ contains a nonzero simple object Σ which is not 1-faithful. This is a contradiction. Hence $(gr^C)_1 = 0$ and Proposition 4.3 yields that C is strongly graded.

4. A GRADED CLIFFORD THEORY FOR COALGEBRAS

Let $C = \oplus_{\sigma \in G} C_\sigma$ be a G-graded coalgebra. We consider the forgetful functor $U: gr^C \to \mathbf{M}^C$. In this section we study the structure of the object $U(\Sigma)$, when Σ is a simple object in gr^C, i.e., we give the structure of the C-comodule Σ considered without the grading, when Σ is a graded simple C-comodule.

In this section we use extensively the results from [7] where the structure of graded simple R-modules is given for the case R a graded ring. First, it is necessary to make some general considerations. Let $C = \oplus_{\sigma \in G} C_\sigma$ be a G-graded coalgebra and let $R = \oplus_{\lambda \in G} R_\lambda$ be the graded ring associated to the coalgebra C (see Sect. 1). We consider the functor $H: gr^C \to R - gr$, $H(M) = \overline{M}$ for any $M \in gr^C$. Let C^* be the dual ring associated to C. The next result establishes that R-homomorphisms and C^*-homomorphisms between two comodules coincide.

PROPOSITION 4.1. *Let $M, N \in gr^C$. The following assertions hold*:

(i) *If $m \in M$ is a homogeneous element and $f \in C^*$, there exists $r \in R$ such that $fm = rm$.*

(ii) $\operatorname{Hom}_R(\overline{M}, \overline{N}) = \operatorname{Hom}_{C^*}(\overline{M}, \overline{N})$.

Proof. (i) Assume that $m \in M_\sigma$, i.e., $\deg m = \sigma$. Since $\rho_M(M_\sigma) \subseteq \sum_{xy=\sigma} M_x \otimes C_y$, then $\rho_M(m) = \sum m_0 \otimes m_1$, where $\deg m_0 \deg m_1 = \sigma$. We have $fm = \sum m_0 \otimes f(m_1)$. Fix $y \in G$. We denote by r_y: $C \to k$ the k-linear map $r_y(c) = f(c)$ if $c \in C_y$ and $r_y(c) = 0$ otherwise. Clearly $r_y \in R_y$. Then we have $r_y m = \sum m_0 r_y(m_1)$, where $\deg m_1 = y$ and $\deg m_0 = \sigma y^{-1}$. In the sum $\rho_M(m) = \sum m_0 \otimes m_1$ the nonzero homogeneous elements m_1 are finite. Let $\{y_1, \dots, y_s\}$ be the set of all the degrees of such elements m_1. If we put $r = r_{y_1} + \cdots + r_{y_s}$, we obtain $fm = rm$ and $r \in R$.

(ii) Since $R \subseteq C^*$, then $\operatorname{Hom}_R(\overline{M}, \overline{N}) \supseteq \operatorname{Hom}_{C^*}(\overline{M}, \overline{N})$. Assume now $u \in \operatorname{Hom}_R(\overline{M}, \overline{N})$. We only need to check that $u(fm) = fu(m)$ for any $f \in C^*$, $m \in M$. Clearly we can assume m homogeneous with degree σ.

If we put $M' = Rm = C^*m$, M' is a graded R-submodule of M. Also since M is a right C-comodule, then C^*m has finite dimension over k. If we consider the restriction of u to M', we denote it by u', then we only have to show that $u'(fm) = fu'(m)$ in order to have the desired equality $u(fm) = fu(m)$. Thus we may assume that M is a graded comodule with finite dimension. In this case it is well known (cf. [10]) that $u \in \operatorname{Hom}_R(\overline{M}, \overline{N})$ admits a decomposition $u = u_{\sigma_1} + \cdots + u_{\sigma_s}$ where u_{σ_i} is a homogeneous morphism of degree σ_i $(1 \leq i \leq n)$. We can restrict to the case when $u \in \operatorname{Hom}_R(\overline{M}, \overline{N})$ is homogeneous of degree $\theta \in G$. If we change N by $N(\sigma)$, then u is a morphism of degree 1, and we have $u \in \operatorname{Hom}_{R-gr}(\overline{M}, \overline{N})$, i.e., $u \in \operatorname{Com}_{gr^C}(M, N)$.

Since $\rho_N(u(m)) = \sum u(m_0) \otimes m_1$ and using an argument similar to the proof of assertion (i), we find $r \in R$ such that $fm = rm$ and $fu(m) = ru(m)$. Thus $u(fm) = u(rm) = ru(m) = fu(m)$. This finishes the proof.

If $M \in gr^C$, we denote $G\{M\} = \{\sigma \in G \mid M \cong M(\sigma) \text{ in } gr^C\}$. Clearly $G\{M\}$ is a subgroup of G. If $M \in gr^C$ and $\dim_k M < \infty$, then $G\{M\}$ is a finite subgroup of G. Indeed, since $\dim_k M < \infty$, there exist $\sigma_1, \dots, \sigma_s \in G$

such that $M_\sigma = 0$ for any $\sigma \notin \{\sigma_1, \dots, \sigma_s\}$. Now if $\sigma \in G\{M\}$, then $M \cong M(\sigma)$. Hence $\{\sigma_1, \dots, \sigma_s\} = \{\sigma\sigma_1, \dots, \sigma\sigma_s\}$. Thus $\sigma \in \{\sigma_i\sigma_j^{-1} \mid 1 \le i, j \le s\}$. Therefore $G\{M\}$ is finite. In particular when Σ is gr-simple, it follows that $G\{\Sigma\}$ is a finite subgroup of G.

Let $\mathscr{A}$ be a Grothendieck category and $M \in \mathscr{A}$. We denote by $(\mathscr{A}|M)$ the full subcategory of $\mathscr{A}$ whose objects are M-generated, i.e., they are homomorphic images of a direct sum of copies of M. In the case when $\mathscr{A} = R - \text{Mod}$, where R is an arbitrary ring, we denote $(R|M) - \text{Mod} = (\mathscr{A}|M)$. When $\mathscr{A} = \mathbf{M}^C$ for C a coalgebra and $M \in \mathbf{M}^C$, we put $\mathbf{M}^{(C|M)} = (\mathbf{M}^C|M)$. By Proposition 4.1, if C is a graded coalgebra and $M \in gr^C$, then we have the equality $\mathbf{M}^{(C|M)} = (R|M) - \text{Mod} = (C^*|M) - \text{Mod}$ where R is the graded ring associated to C.

Now we phrase the main result of this section.

THEOREM 4.2. *Let $C = \oplus_{\sigma \in G} C_\sigma$ be a G-graded coalgebra and let $\Sigma = \oplus_{\lambda \in G} \Sigma_\lambda$ be a right graded simple C-comodule. We denote $\Delta = \text{End}_{C^*}(\Sigma)$. Then the following assertions hold*:

(i) *The category $(C^*|\Sigma) - \text{Mod}$ is equivalent with the category Δ-Mod.*

(ii) *The ring Δ is a $G\{\Sigma\}$-crossed product. In fact Δ is a quasi-Frobenius k-algebra of finite dimension.*

(iii) *If $n = |G\{\Sigma\}|$ and Σ is n-torsionfree, then Σ is a semisimple object of finite length in $C^* - \text{Mod}$, i.e., Σ is a semisimple comodule in $\mathbf{M}^C$.*

(iv) *If the group G is torsionfree, then Σ is a simple C^*-module, i.e., Σ is a simple comodule in $\mathbf{M}^C$.*

Proof. By Proposition 4.1 we have $(C^*|\overline{\Sigma}) - \text{Mod} = (R|\overline{\Sigma}) - \text{Mod}$. Since Σ is a right graded simple comodule, then $\overline{\Sigma}$ is a left simple graded R-module. Now we can apply [7, Theorems 2.10 and 3.2] and the previous results.

COROLLARY 4.3. *Let C be a G-graded k-coalgebra. If* char $k = 0$, *then every gr-simple comodule is semisimple C-comodule of finite length.*

Proof. We apply Theorem 4.2, assertion (iii).

We now describe some applications to the coradical filtrations. We first recall some elementary facts about coradicals (see [13]). For any coalgebra C, the coradical C_0 of C is the sum of all simple subcoalgebras of C. Starting from C_0 the coradical filtration of C can be defined as follows (cf. [13])

$$C_n = \Delta^{-1}(C \otimes C_0 + C_{n-1} \otimes C) = \Delta^{-1}(C \otimes C_{n-1} + C_0 \otimes C).$$

By [13, Corollary 9.1.7] we have $\Delta(C_n) = \sum_{i=0}^{n} C_i \otimes C_{n-i}$. In particular C_n are subcoalgebras of C.

As we have seen C is a (C^*, C^*)-bimodule, where C^* is the dual ring. We note that C_0 is exactly the left (or right) socle $s_0({}_{C^*}C)$ (or $s_0(C_{C^*})$).

Now, if $C = \oplus_{\sigma \in G} C_\sigma$ is a G-graded coalgebra, the graded coradical filtration of C is defined in the following way: C_0^g is the sum of all graded simple subcoalgebras of C and for $n \geq 1$, we establish

$$C_n^g = \Delta^{-1}(C \otimes C_0^g + C_{n-1}^g \otimes C) = \Delta^{-1}(C \otimes C_{n-1}^g + C_0^g \otimes C).$$

If we consider the graded ring R associated to C, then $\overline{C} = H(C)$ is a graded $(R - R)$-bimodule. In this case C_0^g is exactly the left graded socle $s_0^g({}_R\overline{C})$ (or the right graded socle $s_0^g(\overline{C}_R)$), where $\overline{C}$ is considered as a graded left (resp. right) R-module.

PROPOSITION 4.4. *Let $C = \oplus_{\sigma \in G} C_\sigma$ be a G-graded coalgebra. For any $n \geq 0$, $C_n \subseteq C_n^g$.*

Proof. Recall the following well-known fact in the theory of lattices [12, Proposition III.6.7]: if L is a modular and compactly generated lattice, then the join of all the atoms of L (i.e., the socle of L) is equal to the meet of all essential elements of L; we have that $C_0 = \bigcap\{X | X$ is essential submodule in ${}_{C^*}C\}$ and $C_0^g = \bigcap\{Y | Y$ is gr-essential submodule in ${}_R C\}$. Since, by [10], the property of being gr-essential implies the property of being essential considered without graduation, Proposition 4.1 implies that $C_0 \subseteq C_0^g$. The rest of the statement follows by induction.

COROLLARY 4.5. *Let $C = \oplus_{\sigma \in G} C_\sigma$ be a G-graded coalgebra. If either the group G is torsionfree or the* char $k = 0$, *then C_n are graded subcoalgebras of C for any $n \geq 0$.*

Proof. By Theorem 4.2 or Corollary 4.3 we have $C_0^g \subseteq C_0$. Thus $C_0 = C_0^g$. Now by induction we obtain that $C_n = C_n^g$.

Remark. Assume that $C = \oplus_{n \in \mathcal{Z}} C_n$ is a $\mathcal{Z}$-graded coalgebra. Let $\mathcal{R}$ denote the coradical of C. If $C_n = 0$ for any $n < 0$, then $\mathcal{R} \subseteq C_0$. Indeed by Corollary 4.5 it is sufficient to show that if K is a graded simple coalgebra of C, then $K \subseteq C_0$. Now, $K = K_0 \oplus K_1 \oplus \dots$ and $K_0 \neq 0$ (otherwise, $K_0 = 0$ implies $\varepsilon(K) = 0$ since $\varepsilon(K_i) = 0$ for $i \geq 1$; contradiction). Since $\Delta(K_0) \subseteq K_0 \otimes K_0$ then K_0 is a subcoalgebra. Thus K_0 is a gr-subcoalgebra with trivial grading. Since K is gr-simple, then $K = K_0$. This result is given in [13].

The section ends with some applications to gradable modules. If $M \in \mathbf{M}^C$ is a right C-comodule such that M belongs to the image of functor U, the forgetful functor, then M is called a *gradable* module.

COROLLARY 4.6. *Let $C = \oplus_{\sigma \in G} C_\sigma$ be a G-graded coalgebra. The following assertions hold*:

(i) *For every right simple comodule S in $\mathbf{M}^C$, there exists a simple object Σ in the category gr^C such that S is isomorphic to a C-submodule of Σ.*

(ii) *If the group G is torsionfree, then every right simple C-comodule is gradable.*

(iii) *If S is a right simple comodule of $\mathbf{M}^C$, then S is a semisimple of finite length C_1-comodule.*

Proof. (i) Let $\rho_S: S \to S \otimes C$ be the structure map of S as C-comodule. Since $(\varepsilon \otimes 1) \circ \rho_S = 1_S$, then ρ_S is a monomorphism. Let $\pi: C \to C_1$ be the canonical coalgebra map. Since the functor $(-)_\pi$ is a left adjoint to the functor $(-)^\pi = - \square_{C_1} C$, then for every $M \in \mathbf{M}^C$, we have the canonical functorial morphism $\alpha(M): M \to M_\pi \square_{C_1} C$. Since $\alpha(M)$ is the corestriction of ρ_M, then $\rho_M(M) \subseteq M_\pi \square_{C_1} C$ and $\alpha(M)$ is an injective map.

If now $M = S$ is a simple object in $\mathbf{M}^C$, then we have the monomorphism $\alpha(S)$: $S \to S_\pi \square_{C_1} C$. But $S_\pi \square_{C_1} C$ has the grading structure: $S_\pi \square_{C_1} C = \oplus_{\sigma \in G} S_\pi \square_{C_1} C_\sigma$. Hence there exists an object $Y \in gr^C$ and a monomorphism of comodules $0 \to S \to Y$. Let $s_0^g(Y)$ be the socle of Y in the category gr^C. By [12, Chap. III, Proposition 6.7], $s_0^g(Y)$ is the intersection of all essential subobjects of Y in the category gr^C. Thus $s_0^g(Y) = \bigcap\{X | X$ essential in $Y\}$. Being X essential in Y in the category gr^C, it follows that X is also essential in Y in the category $\mathbf{M}^C$. Since S is nonzero, then $S \cap X \neq 0$ for every essential submodule X of Y. Hence $S \subseteq X$ and therefore $S \subseteq s_0^g(Y)$.

On the other hand $s_0^g(Y)$ is a semisimple object in the category gr^C. We can write $s_0^g(Y) = \oplus_{i \in I} \Sigma_i$, where Σ_i are simple objects in the category gr^C. Thus we have the monomorphism $0 \to S \to \oplus_{i \in I} \Sigma_i$. Since S is a simple object in $\mathbf{M}^C$, there exists an $i_0 \in I$ such that S is isomorphic to a submodule of Σ_{i_0}.

(ii) Follows from Theorem 4.2, assertion (iv).

(iii) By assertion (i), there exists a simple object $\Sigma = \oplus_{\lambda \in G} \Sigma_\lambda$ in the category gr^C such that S is isomorphic to a C-subcomodule of Σ. By Theorem 3.6, Σ_λ is either zero or a simple C_1-comodule. Thus Σ is a semisimple C_1-comodule. Clearly S is also semisimple. But S is finite dimensional as k-space, so it follows that S has finite length.

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REFERENCES

1. E. ABE, "Hopf Algebras," Cambridge Univ. Press, Cambridge, UK, 1977.
2. E. DADE, Clifford theory for group-graded rings, *J. Reine Angew. Math.* **369** (1986), 40–86.
3. E. DADE, Clifford theory for group-graded rings, II, *J. Reine Angew. Math.* **387** (1988), 148–181.
4. S. DASCĂLESCU, C. NĂSTĂSESCU, S. RAIANU, AND F. VAN OYSTAEYEN, Graded coalgebras and Morita–Takeuchi context, preprint.
5. Y. DOI, Homological coalgebra, *J. Math. Soc. Japan* **33** (1981), 31–50.
6. P. GABRIEL, Des catégories abeliennes, *Bull. Soc. Math. France* **90** (1962), 323–448.
7. J. L. GÓMEZ PARDO AND C. NĂSTĂSESCU, Relative projectivity, graded Clifford theory, and applications, *J. Algebra* **141** (1991), 484–504.
8. B. MITCHELL, "Theory of Categories," Academic Press, Pure and Appl. Math., San Diego, 1965.
9. C. NĂSTĂSESCU, Some constructions over graded rings: Applications, *J. Algebra* **120** (1989), 119–138.
10. C. NĂSTĂSESCU AND F. VAN OYSTAEYEN, "Graded Ring Theory," North-Holland, Amsterdam, 1982.
11. C. NĂSTĂSESCU AND B. TORRECILLAS, Graded coalgebras, *Tsukuba J. Math.* **17** (1993), 461–479.
12. B. STENSTRÖM, "Rings of Quotients," Springer-Verlag, Berlin, 1975.
13. M. E. SWEEDLER, "Hopf Algebras," Benjamin, New York, 1969.